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Numerical Solution of Generalized Logarithmic Integral Equations of the Second Kind by Projections

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ABSTRACT

In this work, we present a new techniques to solve the integral equations of the second kind with logarithmic kernel. First, we show the existence and uniqueness of the solution for the given problem in a Hilbert space. Next, we discuss a projection method for solving integral equations with logarithmic kernel of the second kind; the present method based on the shifted Legendre polynomials. We examine the existence of the solution for the approximate equation, and we provide a new error estimate for the numerical solutions. At the end, numerical examples are provided to illustrate the theoretical results.

Keywords: Logarithmic kernel, integral equations, projection approximations, shifted Legendre polynomials.

1. Introduction

In past years, several problems of mathematics, engineering, physics and related sciences are described in terms of integral equations, specifically, singular integral equations (cf. Ladopoulos (2001)). An important class of these equations is the singular integral equations with logarithmic kernel. Projection approximation methods play an important role in numerical analysis, especially the projection method is an effective means of numerical solution of integral equations (cf.Atkinson (1997), Kress (1998)).

Mennouni (2012) presented a projection method for solving operator equations with bounded operator in Hilbert spaces, and he applied the method for solving the Cauchy integral equations for two cases: Galerkin projections and Kulkarni projections respectively, using a sequence of orthogonal finite rank projections.

Mennouni (2011) introduced a modified method which is based on the trapezoidal and Simpson's rules, for solving a Volterra integral equations of the second kind. Mennouni and Guedjiba (2010) discussed a projection method for solving Cauchy integro-differential equations via airfoil polynomials of the first kind.

Shestopalov and Chernokozhin (1977) discussed the solutions of boundary integral equations and systems of logarithmic integral equations of the first kind. Banerjea and Sarkar (2006) solved a logarithmic singular integral equation in two disjoint finite intervals by using function theoretic method, under some conditions. Chakrabarti and Sahoo (1996) considered a direct method to solve a singular integral equations of the first kind, involving the combination of a logarithmic and a Cauchy type singularity. Bruckner (1995) solved an integral equation of the first kind with logarithmic kernel as an improperly posed problem.

Morland (1970) presented a closed form solutions for an important class of difference singular integral equations of the first kind. He considered the kernel as the sum of a polynomials, the second one is multiplied by a logarithm. MacCamy (1958) introduced a method to solve a singular integral equations with logarithmic or Cauchy kernels.

Tang and McKee (1992) investigated the analytical properties of a logarithmic singular integral equation. He developed two product integration methods to solve this class of integral equation, the first one based on Euler's method, but in the second one the author used a product trapezoidal method. Panigrahi and Nelakanti (2013) considered the Galerkin method for singular Fredholm integral equations of the second kind with weakly kernel and its corresponding eigenvalue problem on $[-1, 1]$. More recently, Assari and Dehghan (2017) described a collocation method using the radial basis function to numerically solve a boundary singular integral equations of the second kind with logarithmic kernels. These class of integral equations obtained from boundary value problems of Laplace equations with linear Robin boundary conditions.

Recently many researchers developed numerical methods that solve integral equations of the first kind with logarithmic kernel via collocation and Galerkin methods, see Christiansen (1971, 1982), Hsiao and MacCamy (1973), Hsiao and Wendland (1977), Sloan and Spence (1988a,b) and references therein. Based on the above, integral equations with logarithmic kernel are an important type of singular integral equations.

This class of singular integral equations has an important applications in several problems of economics, fluid dynamics, electrodynamics, elasticity, fracture mechanics, biology and other scientific fields and the latest high technology.

The goal of the present paper is to introduce a shifted Legendre projection method to solve generalized integral equations with logarithmic kernel of the second kind. We use new techniques to show the existence and uniqueness of the solution for the present problem in Hilbert spaces and the solution of its corresponding approximate problem.

2. Existence and uniqueness of solutions

Let H be a Hilbert space. $BL(H)$ will denote the space of bounded linear operators from H into itself, and sp denotes the spectrum.
Let $T \in BL(H)$, and let T^* be the adjoint of T. We recall that T is selfadjoint if $T^* = T$, and that T is skew-Hermitian if $T^* = -T$.

Lemma 2.1. *Let $T \in BL(H)$.*

1. *If T is self-adjoint, then $sp\,(T) \subseteq \mathbb{R}$.*
2. *If T is skew-Hermitian, then $sp\,(T) \subseteq \mathrm{i}\mathbb{R}$.*

Proof.

1. See Porter and Stirling (1990).

2. It is clear that
$$(\mathrm{i}T)^* = -\mathrm{i}T^* = \mathrm{i}T,$$
so that the operator $\mathrm{i}T$ is self-adjoint, say $sp\ (\mathrm{i}T) \subseteq \mathbb{R}$, hence $sp\ (T) \subseteq \mathrm{i}\mathbb{R}$.

□

Throughout our paper, denote by $\mathcal{H} := L^2([0,1], \mathbb{C})$. Let us consider the generalized integral equation with logarithmic kernel

$$\int_0^1 h(s,\varsigma) \ln|\varsigma - s|\, \varphi(\varsigma) d\varsigma = \lambda\varphi(s) + f(s), \quad 0 \le s \le 1. \tag{1}$$

We examine the numerical solution of this equation. Our discussion will be in two important cases.

2.1 Case $\overline{h(s,\varsigma)} = -h(\varsigma, s)$

In the first case, we assume that λ is real and non-zero, and $h(.,.)$ is continuous function, moreover,
$$\overline{h(s,\varsigma)} = -h(\varsigma, s).$$
Letting
$$Su(s) \quad := \quad \int_0^1 h(s,\varsigma) \ln|\varsigma - s|\, u(\varsigma) d\varsigma. \quad u \in \mathcal{H}, \quad 0 \le s \le 1.$$

We recall that $S \in BL(\mathcal{H})$, further $S^* = -S$.

Equation (1) is equivalent to
$$(S - \lambda I)\varphi = f.$$

Theorem 2.1. *For all $f \in \mathcal{H}$, the logarithmic integral equation (1) has a unique solution $\varphi \in \mathcal{H}$.*

Proof. Since
$$\overline{h(s,\varsigma)} = -h(\varsigma, s),$$
it follows that S is skew-Hermitian operator, and by Lemma 2.1, we get $sp\ (S) \subseteq \mathrm{i}\mathbb{R}$. This shows that $\lambda \notin sp\ (S)$, consequently the operator $S - \lambda I$ is invertible. □

Theorem 2.2. *The following estimate holds:*

$$\|(S-\lambda I)^{-1}\| \leq \frac{1}{|\lambda|}.$$

Proof. As in Porter and Stirling (1990), for all $u \in \mathcal{H}$,

$$\begin{aligned} Re\,\langle (S-\lambda I)\,u,u\rangle &= \frac{1}{2}\left[\langle (S-\lambda I)\,u,u\rangle + \overline{\langle (S-\lambda I)\,u,u\rangle}\right] \\ &= \frac{1}{2}\left[-2\lambda\,\langle u,u\rangle + \langle Su,u\rangle + \langle u,Su\rangle\right] \\ &= -\lambda\,\langle u,u\rangle\,, \end{aligned}$$

so that

$$|\lambda|\,\|u\|^2 = |\mathrm{Re}\,\langle (S-\lambda I)u,u\rangle| \leq |\langle (S-\lambda I)u,u\rangle| \leq \|(S-\lambda I)u\|\,\|u\|\,,$$

which yields

$$\|(S-\lambda I)^{-1}\| \leq \frac{1}{|\lambda|}.$$

□

2.2 Case $\overline{h(s,\varsigma)} = h(\varsigma,s)$

In this case, we assume that $\lambda \in i\mathbb{R}^*$ and $h(.,.)$ satisfies

$$\overline{h(s,\varsigma)} = h(\varsigma,s),$$

hence

$$S^* = S.$$

Theorem 2.3. *For all $f \in \mathcal{H}$, the logarithmic integral equation (1) has a unique solution $\varphi \in \mathcal{H}$.*

Proof. Since

$$\overline{h(s,\varsigma)} = h(\varsigma,s),$$

we deduce that S is selfadjoint operator. Hence

$$sp\,(S) \subseteq \mathbb{R},$$

and hence $\lambda \notin sp\,(S)$, which proves that the operator $S - \lambda I$ is invertible. □

Theorem 2.4. *The following estimate holds:*

$$\|(S-\lambda I)^{-1}\| \leq \frac{1}{|\mathrm{Im}(\lambda)|}.$$

Proof. As in Porter and Stirling (1990), for all $u \in \mathcal{H}$, we have

$$\begin{aligned} |\langle (S-\lambda I)\, u, u\rangle|^2 &= \left|\langle Su, u\rangle - \lambda \|u\|^2\right|^2 \\ &= \left|\langle S, u\rangle - \mathrm{Re}(\lambda)\|u\|^2\right|^2 + |\mathrm{Im}(\lambda)|^2 \|u\|^4 \\ &\geq |\mathrm{Im}(\lambda)|^2 \|u\|^4, \end{aligned}$$

so that

$$|\mathrm{Im}(\lambda)|\, \|u\|^2 \leq |\langle (S-\lambda I)\, u, u\rangle|,$$

which yields

$$\|(S-\lambda I)^{-1}\| \leq \frac{1}{|\mathrm{Im}(\lambda)|}.$$

□

3. Bounded finite rank orthogonal projections

Let $(L_n)_{n\geq 0}$ denote the sequence of Legendre polynomials. We recall that the Legendre polynomials $L_n(.)$ can be defined by

$$L_n(s) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \left(\frac{1+s}{2}\right)^{n-k} \left(\frac{1-s}{2}\right)^k.$$

Their generating function is given by

$$\frac{1}{\sqrt{1-2st+t^2}} = \sum_{n=0}^{\infty} L_n(s) t^n.$$

Note that the first few Legendre polynomials are

$$\begin{aligned} L_0(s) &:= 1; \\ L_1(s) &:= s; \\ L_2(s) &:= \frac{1}{2}\left(3s^2-1\right); \\ L_3(s) &:= \frac{1}{2}\left(5s^3-3s\right); \\ L_4(s) &:= \frac{1}{8}\left(35s^4-30s^2+3\right). \end{aligned}$$

The Legendre polynomials of higher degrees are often found by employing a three-term recurence relation (see Phillips (2003)).

Let us consider the following shifted Legendre polynomials

$$\ell_n(s) := L_n(2s-1), \quad 0 \leq s \leq 1.$$

Recall that the shifted Legendre polynomials are orthogonal on $[0,1]$.

An explicit formula for the shifted Legendre polynomials is given by

$$\ell_n(s) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-s)^k.$$

The first few shifted Legendre polynomials are:

$$\begin{aligned} \ell_0(s) &:= 1; \\ \ell_1(s) &:= 2s-1; \\ \ell_2(s) &:= 6s^2-6s+1; \\ \ell_3(s) &:= 20s^3-30s^2+12s-1; \\ \ell_4(s) &:= 70s^4-140s^3+90s^2-20s+1. \end{aligned}$$

Letting

$$e_j(t) := \frac{1}{\sqrt{2j+1}} \ell_j(t), \quad 0 \leq t \leq 1.$$

We define the space $\mathcal{H}_n$ spanned by $\{e_j, \quad j = 0 \ldots n\}$. We associate to $\mathcal{H}_n$ the sequence $(\pi_n)_{n \geq 0}$ of bounded finite rank orthogonal projections onto $\mathcal{H}_n$ given by

$$\pi_n u := \sum_{j=0}^{n} \langle u, e_j \rangle \, e_j.$$

We recall that, for all $\psi \in \mathcal{H}$,

$$\lim_{n \to \infty} \|\pi_n \psi - \psi\| = 0.$$

We consider the following approximate operator $S_n := \pi_n S \pi_n$.

3.1 Case $\overline{h(s,\varsigma)} = -h(\varsigma, s)$

Since S is skew-Hermitian operator, it follows that $(S_n)_{n \geq 1}$ is a sequence of skew-Hermitian operator from $\mathcal{H}$ into itself.

Theorem 3.1. *For all n, the operator $S_n - \lambda I$ is invertible, and*

$$\| (S_n - \lambda I)^{-1} \| \leq \frac{1}{|\lambda|}.$$

Proof. Obviously that $\lambda \notin sp\ (S_n)$ (cf. Lemma 2.1), and hence the operator $S_n - \lambda I$ is invertible. We easily obtain

$$Re \langle (S_n - \lambda I)\, u, u \rangle \quad = \quad -\lambda \langle u, u \rangle ,$$

so that

$$|\lambda| \langle u, u \rangle \leq |\langle (S_n - \lambda I)\, u, u \rangle| \leq \|(S_n - \lambda I)\, u\| \, \|u\| ,$$

that is to say

$$\|(S_n - \lambda I)\, u\| \geq |\lambda| \, \|u\| .$$

Clearly, we have the bound

$$\| (S_n - \lambda I)^{-1} \| \leq \frac{1}{|\lambda|}.$$

□

3.2 Case $\overline{h(s,\varsigma)} = h(\varsigma, s)$

Since S is selfadjoint operator, it follows that $(S_n)_{n\geq 1}$ is a sequences of selfadjoint operators from $\mathcal{H}$ into itself.

Theorem 3.2. *For all n, the operator $S_n - \lambda I$ is invertible, moreover*

$$\| (S_n - \lambda I)^{-1} \| \leq \frac{1}{|\mathrm{Im}(\lambda)|}.$$

Proof. As before, for all $u \in \mathcal{H}$, we have

$$\begin{aligned} |\langle (S_n - \lambda I)\, u, u \rangle|^2 &= \left| \langle S_n u, u \rangle - \lambda \|u\|^2 \right|^2 \\ &= \left| \langle S_n u, u \rangle - \mathrm{Re}(\lambda) \|u\|^2 \right|^2 + |\mathrm{Im}(\lambda)|^2 \|u\|^4 \\ &\geq |\mathrm{Im}(\lambda)|^2 \|u\|^4 , \end{aligned}$$

hence

$$|\mathrm{Im}(\lambda)| \, \|u\|^2 \leq |\langle (S_n - \lambda I)\, u, u \rangle| .$$

This leads to

$$\|(S_n - \lambda I)^{-1}\| \leq \frac{1}{|\mathrm{Im}(\lambda)|}.$$

□

4. Approximate solutions

The approximate problem is the following equation for φ_n:

$$S_n\varphi_n - \lambda\varphi_n = \pi_n f, \tag{2}$$

For all n, the approximate equation (2) has a unique solution φ_n, given by

$$\varphi_n = \sum_{j=0}^{n} c_j e_j,$$

for some scalars c_j. Equation (2) reads as

$$\sum_{j=0}^{n} c_j \left[\pi_n S e_j - \lambda e_j\right] = \pi_n f,$$

so that

$$\sum_{j=0}^{n} c_j \left[\sum_{i=0}^{n} \langle Se_j, e_i\rangle e_i - \lambda e_j\right] = \sum_{i=0}^{n} \langle f, e_i\rangle e_i,$$

that is to say, the coefficients c_j are obtained by solving the following linear system

$$(A_n - \lambda I)x_n = b_n, \tag{3}$$

where

$$\begin{aligned} A_n(k,j) &:= \int_0^1 \int_0^1 h(s,\varsigma) \ln|\varsigma - s|\, e_j(\varsigma) e_k(s) d\varsigma ds, \\ b_n(k) &:= \int_0^1 e_k(s) f(s) ds. \end{aligned}$$

As in Atkinson (1997), Fang et al. (2004), we introduce a quadrature rule for computing $A_n(k,j)$.

Let us consider the following change of variables

$$\tau := \varsigma - s, \quad \upsilon := \varsigma + s,$$

that is

$$\varsigma = \frac{\tau + \upsilon}{2}, \quad s = \frac{-\tau + \upsilon}{2}.$$

Letting

$$V(\tau,\upsilon) := \frac{1}{2}h(\frac{-\tau+\upsilon}{2},\frac{\tau+\upsilon}{2})\ln|\tau|\, e_j(\frac{\tau+\upsilon}{2})e_k(\frac{-\tau+\upsilon}{2}),$$

$$\alpha(\tau) := \max(-\tau,\tau), \quad \beta(\tau) := \min(2-\tau, 2+\tau),$$

and

$$\vartheta(\tau) := \int_{\alpha(\tau)}^{\beta(\tau)} V(\tau,\upsilon)d\upsilon,$$

we get

$$A_n(k,j) := \int_{-1}^{1} \vartheta(\tau)d\tau.$$

Consider the following partition points of the interval $[-1,1]$:

$$-1+s_j, \quad -s_j, \quad s_j, \quad 1-s_j, \quad j=0,1,\ldots,m,$$

with

$$s_j = \frac{1}{2}(\frac{j}{m})^q, \quad j=0,1,\ldots,m.$$

Hence

$$A_n(k,j) = \sum_{i=1}^{4m}\int_{\tau_{i-1}}^{\tau_i} \vartheta(\tau)d\tau,$$

where $\tau_0,\tau_1,\ldots,\tau_{4m}$ are partition points in ascending order.

Letting

$$G(t) := \sum_{i=1}^{4m}\frac{\tau_i-\tau_{i-1}}{2}\vartheta(\frac{\tau_i-\tau_{i-1}}{2}t+\frac{\tau_i-\tau_{i-1}}{2}),$$

we obtain

$$A_n(k,j) = \int_0^1 G(t)dt.$$

In order to compute $b_n(k)$ accurately and effectively, we use the Legendre Gauss Lobatto quadrature (cf. Canuto et al. (2006), pp. 331).

4.1 Convergence Analysis

Let $\rho > 0$ and $H^{\rho}(0,1)$ be the classical Sobolev space, and let $\|.\|_{\rho}$ denotes its norm. (Cf. Atkinson and Han (2009), pp. 119.)

As in Atkinson and Han (2009) there exists $c > 0$ such that, for all $x \in H^\rho([0,1],\mathbb{C})$,

$$\|(I-\pi_n)x\| \leq cn^{-\rho}\|x\|_\rho. \tag{4}$$

Theorem 4.1. *Assume that* $f \in H^\rho([0,1],\mathbb{C})$, *and*

$$\overline{h(s,\varsigma)} = -h(\varsigma,s).$$

There exists a positive constant c, such that:

$$\|\varphi_n - \varphi\| \leq \frac{cn^{-\rho}}{|\lambda|}\left[(1+2\|S\|)\|\varphi\|_\rho + \|f\|_\rho\right],$$

for n large enough.

Proof. We have

$$\begin{aligned}
\varphi_n - \varphi &= (S_n-\lambda I)^{-1}\pi_n f - (S-\lambda I)^{-1}f \\
&= (S_n-\lambda I)^{-1}\pi_n f - (S_n-\lambda I)^{-1}f + (S_n-\lambda I)^{-1}f - (S-\lambda I)^{-1}f \\
&= (S_n-\lambda I)^{-1}(\pi_n - I)f + (S_n-\lambda I)^{-1}\left[(S-\lambda I)-(S_n-\lambda I)\right](S-\lambda I)^{-1}f \\
&= (S_n-\lambda I)^{-1}\left[(\pi_n-I)f + (S-S_n)\varphi\right].
\end{aligned}$$

But

$$\left\|(S_n-\lambda I)^{-1}\right\| \leq \frac{1}{|\lambda|},$$

and

$$(S-S_n)\varphi = (I-\pi_n)S\varphi + \pi_n S(\pi_n - I)\varphi,$$

and since $\|\pi_n\| = 1$, then using (4), we get the desired result. □

Theorem 4.2. *Assume that* $f \in H^\rho([0,1],\mathbb{C})$, *and*

$$\overline{h(s,\varsigma)} = h(\varsigma,s).$$

There exists a positive constant c, such that:

$$\|\varphi_n - \varphi\| \leq \frac{cn^{-\rho}}{|\mathrm{Im}(\lambda)|}\left[(1+2\|S\|)\|\varphi\|_\rho + \|f\|_\rho\right],$$

for n large enough.

Proof. Proceed in the similar manner as above, and using

$$\left\|(S_n-\lambda I)^{-1}\right\| \leq \frac{1}{|\mathrm{Im}(\lambda)|},$$

we get the desired result. □

4.2 Classical Fredholm integral equation with logarithmic kernel of the second kind

In this section, we turn our attention to the following classical Fredholm integral equation with logarithmic kernel

$$\int_0^1 \ln|\varsigma - s|\,\varphi(\varsigma)d\varsigma = \lambda\varphi(s) + f(s), \quad 0 \le s \le 1.$$

We assume that this equation has unique solution in $\mathcal{H}$.

Letting

$$Ku(s) \quad := \quad \int_0^1 \ln|\varsigma - s|\,u(\varsigma)d\varsigma, \quad u \in \mathcal{H}, \quad 0 \le s \le 1,$$

$$K_n := \pi_n K \pi_n.$$

We recall that K is compact from $\mathcal{H}$ into itself, (see Atkinson (1997) pp. 8), further, $\|K\| = 1 + \ln 2$, (see Kythe and Puri (2002), pp. 228). It is shown that the inverse operator $(I - K_n)^{-1}$ exists and is uniformly bounded for n large enough, (see Atkinson (1997) pp. 55).

As above, there exists a positive constant c, such that:

$$\|\varphi_n - \varphi\| \le cn^{-\rho}\left[(3 + 2\ln 2)\|\varphi\|_\rho + \|f\|_\rho\right],$$

for n large enough.

5. Numerical Computations

In this section, we present some numerical examples to illustrate the theoretical results obtained in the previous sections. For computations we use Gauss quadrature rule to solve the linear system. The errors of the projection method are presented for different kernels, and for several values of n.

In examples 4 and 5, we compare the present results with previous results presented in many works related to logarithmic singular integral equations. We use the suggested method to solve the particular integral equations in all examples. We evaluate $A_n(k,j)$ and $b_n(k)$.

Once the above matrix equation (3) is solved, we find $x_n := [c_j, \quad j = 0, \cdots, n]$, hence the solution φ_n is built through

$$\varphi_n(t) = \sum_{j=0}^{n} \frac{c_j}{\sqrt{2j+1}} \ell_j(t), \quad 0 \le t \le 1.$$

The main advantages of the present method are that we give a new theoretical framework for the logarithmic singular Fredholm integral equations by projections, the method is applicable even for the particular cases mentioned above with a better accuracy and we obtain new error estimate, which is small in comparison with other methods. However, it may be difficult to use the present method for solving nonlinear integral equations.

Example 1

Let us first consider the generalized integral equation with logarithmic kernel (1) where f is chosen such that the exact solution is

$$\varphi(s) = s(s-1),$$

and

$$h(s, \varsigma) = s^2 - \varsigma^2, \quad \lambda = 2.$$

It is clear that

$$h(\varsigma, s) = -h(s, \varsigma).$$

Table 1: Absolute errors for Example 1

n	$\lVert\varphi - \varphi_n\rVert$
3	4.266×10^{-3}
4	8.575×10^{-4}
5	1.125×10^{-4}
6	4.751×10^{-6}

Table 1 shows the rate of convergence of the method.

Example 2

Let us consider the generalized integral equation with logarithmic kernel (1) where f is chosen such that the exact solution is

$$\varphi(s) = s^2 - s + 1,$$

and

$$h(s, \varsigma) = (s-1)(\varsigma - 1), \quad \lambda = \mathrm{i}.$$

It is clear that $\overline{h(s,\varsigma)} = h(\varsigma, s)$.

Table 2: Absolute errors for Example 2

n	$\|\varphi - \varphi_n\|$
3	1.373×10^{-5}
4	3.735×10^{-6}
5	5.211×10^{-7}
5	6.283×10^{-7}
6	8.305×10^{-8}

We present in Table 2 the corresponding absolute errors for the example 2.

Example 3

Here, we consider the following integral equation with logarithmic kernel

$$u(s) - \int_0^1 \ln|s-t|\, u(\varsigma) d\varsigma = (1 - \ln(s)e^s + \text{Ei }(s) + e^{s-1}\ln(1-s) - \text{Ei }(x-1))e^{-s}.$$

with the exact solution

$$u(s) = e^{-s}.$$

We note that Ei (.) is the exponential integral function, which is defined as:

$$\text{Ei }(s) := \int_{-s}^{\infty} \frac{e^{-t}}{t} dt.$$

Table 3: Absolute errors for Example 3

n	$\|\varphi - \varphi_n\|_2$
3	1.94539×10^{-4}
4	1.22031×10^{-4}
5	6.32020×10^{-5}
6	1.47377×10^{-5}
7	8.57667×10^{-6}

The corresponding absolute errors for the example 3 are presented in Table 3.

Example 4: (Cf. Baker (1978), pp. 536)

In Baker (1978), the author has used the modified quadrature method and the repeated Simpson's rule with step $h := \frac{1}{n}$ to approximate the solution of the following Fredholm integral equation with logarithmic kernel of the second kind

$$u(s) - \int_0^1 \ln|s-t|\, u(\varsigma)d\varsigma = \frac{3}{2}s - \frac{1}{2}\ln(s)s^2 + \frac{1}{2}\ln(-s+1)s^2 + \frac{1}{4} - \frac{1}{2}\ln(-s+1).$$

with the exact solution

$$u(s) = s.$$

Table 4: Absolute errors for Example 4

s	$\lvert\varphi(s)-\varphi_n(s)\rvert$ in Baker (1978)	$\lvert\varphi(s)-\varphi_n(s)\rvert$ for the present method
0	3.6×10^{-3}	4.9×10^{-10}
0.25	7.5×10^{-5}	$4.\times10^{-10}$
0.5	3.6×10^{-12}	3.1×10^{-13}
0.75	7.5×10^{-5}	0.
1	3.6×10^{-3}	1.4×10^{-10}

As in Baker (1978), we use a computer working to 10 decimal digits. We compare his results with our results for $n = 4$ (see Table (4).

Example 5: (Cf. Atkinson (1997), pp. 117)

The author of Atkinson (1997) has introduced the Nyström method to numerically solve the following integral equation with logarithmic kernel of the second kind

$$u(s) - \int_0^1 \ln|s-t|\, u(\varsigma)d\varsigma = e^s + \ln(s) - e^s\text{Ei}\,(-s) - e^1\ln(1-s) + e^s\text{Ei}\,(1-s),$$

with the exact solution

$$u(s) = e^s.$$

Numerical results of the present method are given in Table 5. In Atkinson (1997), the corresponding uniform norm of the error is 1.16×10^{-3}, for $n = 10$. However, only at $n = 6$, the corresponding uniform norm of error by the present method is 2.1×10^{-5}.

Table 5: Absolute errors for Example 5

n	$\|\varphi - \varphi_n\|_2$
2	6.29377×10^{-3}
3	5.28887×10^{-3}
4	3.31887×10^{-4}
5	1.68863×10^{-5}
6	5.00531×10^{-6}

6. Conclusion

In this paper, we have proposed a projection method to numerically solve generalized Fredholm integral equation with logarithmic kernel of the second kind. For our analysis we used new techniques via spectral theory. The proposed method is based on the shifted Legendre polynomials. We feel that the present method can be used to solve other classes of integral and integro-differential equations.

References

Assari, P. and Dehghan, M. (2017). A meshless discrete collocation method for the numerical solution of singular-logarithmic boundary integral equations utilizing radial basis functions. *Applied Mathematics and Computation*, 315:424–444.

Atkinson, K. (1997). *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge university press, Cambridge.

Atkinson, K. and Han, W. (2009). *Theoretical Numerical Analysis: A Functional Analysis Framework*. Springer, New York.

Baker, C. T. H. (1978). *The Numerical Treatment of Integral Equations*. Oxford University Press, Oxford.

Banerjea, S. and Sarkar, C. (2006). On a singular integral equation with log kernel and its application. *International Journal of Mathematics and Mathematical Sciences*, 2006(ID 35630):12 pages.

Bruckner, G. (1995). On the regularization of the ill-posed logarithmic kernel integral equation of the first kind. *Inverse Problems*, 11:65–71.

Canuto, C., Hussaini, M., Quarteroni, A., and Zang, T. (2006). *Spectral Methods: Fundamentals in Single Domains.* Springer-Verlag, Berlin, Heidelberg.

Chakrabarti, A. and Sahoo, T. (1996). Solution of singular integral equations with logarithmic and Cauchy kernels. *Proc. Indian Acad. (Math. SO.)*, 106:261–270.

Christiansen, S. (1971). Numerical solutions of an integral equation with alogarithmic kernel. *BIT Numerical Mathematics*, 11:276–287.

Christiansen, S. (1982). On two methods for elimination of non-unique solutions of an integral equation with logarithmic kernel. *Applicable Analysis*, 13:1–18.

Fang, W., Wang, Y., and Xu, Y. (2004). An implementation of fast wavelet Galerkin methods for integral equations of the second kind. *Journal of Scientific Computing*, 277-302.

Hsiao, G. and MacCamy, R. (1973). Solution of boundary value problems by integral equations of the first kind. *SIAM Rev.*, 15:687–705.

Hsiao, G. and Wendland, W. (1977). A finite element method for some integral equations of the first kind. *J. Math. Anal. Appl.*, 58:449–481.

Kress, R. (1998). *Linear Integral Equations.* Springer-Verlag, New York.

Kythe, P. K. and Puri, P. (2002). *Computational Methods for Linear Integral Equations.* Birkhäuser, Boston.

Ladopoulos, E. (2001). *Singular Integral Equations.* Springer, New York.

MacCamy, R. (1958). On singular integral equations with logarithmic or Cauchy kernels. *Journal of Mathematics and Mechanics*, 7:355–375.

Mennouni, A. (2011). A new numerical approximation for Volterra integral equations combining two quadrature rules. *Applied Mathematics and Computation*, 218:1962–1969.

Mennouni, A. (2012). Two projection methods for skew-hermitian operator equations. *Mathematical and Computer Modelling*, 55:1649–1654.

Mennouni, A. and Guedjiba, S. (2010). A note on solving integro-differential equation with Cauchy kernel. *Mathematical and Computer Modelling*, 52:1634–1638.

Morland, L. W. (1970). Singular integral equations with logarithmic kernels. *Mathematika*, 17.

Panigrahi, B. L. and Nelakanti, G. (2013). Legendre Galerkin method for weakly singular Fredholm integral equations and the corresponding eigenvalue problem. *Journal of Applied Mathematics and Computing*, 43.

Phillips, G. M. (2003). *Iterpolation and Approximation by Polynomials.* Springer, New York.

Porter, D. and Stirling, D. (1990). *Integral Equations: A Practical Treatment, from Spectral Theory to Applications.* Cambridge University Press, Cambridge.

Shestopalov, Y. and Chernokozhin, E. (1977). On solution to integral equations with a logarithmic singularity of the kernel on several intervals of integration: elements of the spectral theory. *Journal of V. N. Karazin Kharkiv National University*, 1058:184–189.

Sloan, I. and Spence, A. (1988a). The Galerkin method for integral equations of the first kind with logarithmic kernel: Applications. *IMA J. Numer. Anal.*, 8:123–140.

Sloan, I. and Spence, A. (1988b). The Galerkin method for integral equations of the first kind with logarithmic kernel: Theory. *IMA J. Numer. Anal.*, 8:105–122.

Tang, T. and McKee, S. (1992). Product integration methods for an integral equation with logarithmic singular kernel. *Applied Numerical Mathematics*, 9:259–266.